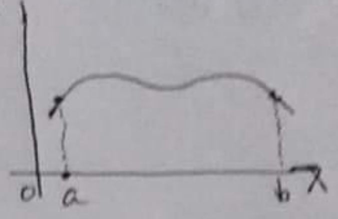
Definite Integral



Definition: The curve (or straight) of a function $f(x)$ is to be integrated from the point $x=a$ to the point $x=b$ is written as $\int_a^b f(x)\,dx$, which is known as Definite integral from a to b.

Exp: Find the value of $\int_0^1 x^3\sqrt{1+3x^4}\,dx$

$$= \frac{1}{12}\int_1^4 \sqrt{z}\,dz$$

put $1+3x^4 = z$

$12x^3\,dx = dz$

$x^3\,dx = \frac{1}{12}dz$

x	0	1
z	1	4

$$= \frac{1}{12}\left[\frac{z^{3/2}}{3/2}\right]_1^4$$

$$= \frac{1}{12}\left[\frac{2}{3}\left(4^{3/2} - 1^{3/2}\right)\right]$$

$$= \frac{1}{12}\cdot\frac{2}{3}(8-1) = \frac{2\cdot 7}{36} = \frac{7}{18} \text{ Ans.}$$

2. Find the value of $\int_0^{\pi}\frac{dx}{2+\cos x}$

$$= \int_0^{\pi}\frac{dx}{2\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$

$$= \int_0^{\pi}\frac{dx}{3\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}$$

$$= \int_0^{\pi}\frac{\sec^2\frac{x}{2}\,dx}{3+\tan^2\frac{x}{2}}$$

put $\tan\frac{x}{2} = z$

$\sec^2\frac{x}{2}\cdot\frac{1}{2}\,dx = dz$

$\sec^2\frac{x}{2}\,dx = 2dz$

x	0	π
z	0	∞

$$= \int_0^{\infty}\frac{2dz}{3+z^2}$$

$$= 2\int_0^\infty \frac{dx}{(\sqrt{3})^2 + x^2} = 2\left[\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right]_0^\infty$$

$$= \frac{2}{\sqrt{3}}(\tan^{-1}\infty - \tan^{-1}0)$$

$$= \frac{2}{\sqrt{3}}\cdot\frac{\pi}{2} = \frac{\pi}{\sqrt{3}} \text{ Ans}$$

Properties of Definite Integral

1. $\int_a^b f(x) = -\int_b^a f(x)\,dx$

2. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$; if $a < c < b$.

3. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

4. $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(2a-x) = f(x)$

   $= 0$ if $f(2a-x) = -f(x)$.

5. $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if f is even i.e $f(-x) = f(x)$

   $= 0$ if f is odd i.e $f(-x) = -f(x)$.

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Beta Function and Gamma Function

Beta Function: $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, where $m,n>0$ is known as First Eulerian Integral or Beta Function.

Gamma Function: $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx$, where $n>0$ is known as Second Eulerian Integral or Gamma Function.

properties:
1. $\Gamma 1 = 1$
2. $\Gamma(n+1) = n\Gamma n = \lfloor n$
3. $B(m,n) = B(n,m)$.
4. $B(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$.
5. $B(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$.
6. $\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$.
7. $\Gamma\frac{1}{2} = \sqrt{\pi}$.

Proof of 2: $\Gamma(n+1) = n\Gamma n = \lfloor n$.

We know that $\Gamma n = \int_0^\infty x^{n-1} e^{-x}\,dx$

$$\Gamma(n+1) = \int_0^\infty x^{n+1-1} e^{-x}\,dx$$

$$\Gamma(n+1) = \int_0^\infty x^{n} e^{-x}\,dx \quad \text{——— (1)}$$

Now $\int x^n e^{-x}\,dx = x^n \int e^{-x}\,dx - \int\left\{\frac{d}{dx}(x^n)\int e^{-x}\,dx\right\}dx$

$$= -x^n e^{-x} - \int\left\{n x^{n-1} \frac{e^{-x}}{-1}\right\}dx$$

$$= -x^n e^{-x} + n\int x^{n-1} e^{-x}\,dx$$

$\therefore \int_0^\infty x^n e^{-x}\,dx = \left[-x^n e^{-x}\right]_0^\infty + n\int_0^\infty x^{n-1} e^{-x}\,dx$

$$-0+0 + n\Gamma n = n\Gamma n \quad \text{Proved.}$$

Again $\Gamma(n+1) = n\Gamma(n)$

$\therefore \Gamma(n) = (n-1)\Gamma(n-1)$

$\Gamma(n-1) = (n-2)\Gamma(n-2)$.

Similarly writing $(n-1)$ ~~for~~ from $(n-2), (n-3)\ldots\ldots$ upto $4\cdot3\cdot2\cdot1$

we get

$$\Gamma(n+1) = n(n-1)(n-2)(n-3)\ldots\ldots 4\cdot3\cdot2\cdot1$$

$$= \lfloor n$$

$\therefore \Gamma(n+1) = n\Gamma(n) = \lfloor n$ proved.

<u>Proof of Property 4.</u> $B(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy$.

we know that $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$,

put $x = \frac{1}{1+y} = (1+y)^{-1}$

then $dx = -(1+y)^{-2}dy$

$= -\frac{1}{(1+y)^2}dy$

$$B(m,n) = -\int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(1-\frac{1}{1+y}\right)^{n-1}\cdot\frac{1}{(1+y)^2}dy$$

$$= \int_0^\infty \frac{1}{(1+y)^{m-1}}\cdot\frac{y^{n-1}}{(1+y)^{n-1}}\cdot\frac{1}{(1+y)^2}dy$$

x	0	1
y	∞	0

$$= \int_0^\infty \frac{1}{(1+y)^{m-1}}\cdot\frac{y^{n-1}}{(1+y)^{n+1}}dy$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy.$$ proved

Relation between Beta Function and Gamma Function

Property 5: $B(m,n) = \dfrac{\Gamma m \, \Gamma n}{\Gamma m+n}$.

Proof :- From the definition, $\Gamma n = \int_0^\infty e^{-x} x^{n-1} \, dx$ —— (1)

$$\therefore \Gamma n = \int_0^\infty e^{-\lambda t} (\lambda t)^{n-1} \lambda \, dt$$

put $x = \lambda t$
$dx = \lambda \, dt$

x	0	∞
t	0	∞

$$= \int_0^\infty e^{-\lambda t} \lambda^{n-1} t^{n-1} \lambda \, dt$$

$$\Gamma n = \int_0^\infty e^{-\lambda t} \lambda^{n} t^{n-1} \, dt \quad \text{—— (2)}$$

Again we can write from the definition

$$\Gamma n = \int_0^\infty e^{-\lambda} \lambda^{n-1} \, d\lambda$$

$$\therefore \Gamma m = \int_0^\infty e^{-\lambda} \lambda^{m-1} \, d\lambda \quad \text{—— (3)}$$

Multiplying (2) & (3), we get

$$\Gamma n \, \Gamma m = \int_0^\infty \int_0^\infty e^{-\lambda t} e^{-\lambda} \lambda^{n} \lambda^{m-1} t^{n-1} \, dt \, d\lambda$$

$$\Gamma n \, \Gamma m = \int_0^\infty \left[\int_0^\infty e^{-\lambda(t+1)} \lambda^{m+n-1} \, d\lambda \right] t^{n-1} \, dt \quad \text{—— (4)}$$

Now from (2), $\dfrac{\Gamma n}{\lambda^n} = \int_0^\infty e^{-\lambda t} t^{n-1} \, dt$

Using this within 3rd bracket of (4),

$$\Gamma n \, \Gamma m = \int_0^\infty \left[\frac{\Gamma m+n}{(t+1)^{m+n}} \right] t^{n-1} \, dt$$

$$= \Gamma m+n \int_0^\infty \frac{t^{n-1}}{(1+t)^{m+n}} \, dt$$

$$\Gamma n \, \Gamma m = \Gamma m+n \cdot B(m,n)$$

[we know that $B(m,n) = \int_0^\infty \frac{y^{n-1} \, dy}{(1+y)^{m+n}}$]

$$\therefore B(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n} \quad \text{Proved.}$$

Problems: 1. Show that $\int_0^{\pi/2} \frac{d\theta}{1+\tan\theta} = \frac{\pi}{4}$

LHS, $I = \int_0^{\pi/2} \frac{d\theta}{1+\tan\theta}$

$$= \int_0^{\pi/2} \frac{d\theta}{1+\frac{\sin\theta}{\cos\theta}}$$

$$= \int_0^{\pi/2} \frac{\cos\theta \, d\theta}{\cos\theta+\sin\theta} \quad \text{------ (i)}$$

$$= \int_0^{\pi/2} \frac{\cos(\pi/2-\theta)\, d\theta}{\cos(\pi/2-\theta)+\sin(\pi/2-\theta)}$$ [Sheet page no 2 Property 3.]

$$I = \int_0^{\pi/2} \frac{\sin\theta \, d\theta}{\sin\theta+\cos\theta} \quad \text{------ (ii)}$$

(i) + (ii) $\Longrightarrow$

$$2I = \int_0^{\pi/2} \frac{\cos\theta \, d\theta}{\sin\theta+\cos\theta} + \int_0^{\pi/2} \frac{\sin\theta \, d\theta}{\sin\theta+\cos\theta}$$

$$= \int_0^{\pi/2} \frac{\cos\theta+\sin\theta}{\sin\theta+\cos\theta}\, d\theta$$

$$= \int_0^{\pi/2} d\theta = [\theta]_0^{\pi/2}$$

$$= \frac{\pi}{2}$$

$\therefore I = \frac{\pi}{4}$ Proved.

Q.2 : Show that $\int_0^1 \frac{\log(1+x)\,dx}{1+x^2} = \frac{\pi}{8}\log 2$.

$$I = \int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$$

Put $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

x	0	1
θ	0	$\pi/4$

$$= \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta}\sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{\sec^2\theta}\cdot\sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta \quad \ldots\ldots (1)$$

$$= \int_0^{\pi/4} \log\left\{1+\tan\left(\pi/4-\theta\right)\right\} d\theta$$

$$= \int_0^{\pi/4} \log\left\{1+\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\cdot\tan\theta}\right\} d\theta$$

$$= \int_0^{\pi/4} \log\left\{1+\frac{1-\tan\theta}{1+\tan\theta}\right\} d\theta$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \log 2\, d\theta - \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$$

$$= \int_0^{\pi/4} \log 2\, d\theta - I \qquad [\text{By } (1)]$$

$$\therefore\ 2I = \int_0^{\pi/4} \log 2\, d\theta$$

$$= \log 2 \int_0^{\pi/4} d\theta = \log 2\,[\theta]_0^{\pi/4}$$

$$= \log 2 \cdot \frac{\pi}{4} = \frac{\pi}{4}\log 2$$

$$\therefore\ I = \frac{\pi}{8}\log 2.$$ Proved.

Q.3: Prove that $\int_0^{\pi/2} \sin^5\theta \cos^4\theta \, d\theta = \frac{8}{315}$.

Let $I = \int_0^{\pi/2} \sin^5\theta \cos^4\theta \, d\theta$

$$= \frac{\Gamma\left(\frac{5+1}{2}\right)\Gamma\left(\frac{4+1}{2}\right)}{2\,\Gamma\left(\frac{5+4+2}{2}\right)}$$

$$= \frac{\Gamma(3)\,\Gamma\left(\frac{5}{2}\right)}{2\,\Gamma\left(\frac{11}{2}\right)}$$

$$= \frac{\Gamma(2)\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)}{2\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{\Gamma(2)}{2\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}} = \frac{\cancel{2}}{\cancel{2}\cdot\frac{315}{8}}$$

$$= \frac{8}{315}$$ Proved.

Using $\Gamma(n+1) = n\Gamma(n)$.

$$\Gamma\left(\frac{11}{2}\right) = \Gamma\left(\frac{9}{2}+1\right)$$
$$= \frac{9}{2}\Gamma\left(\frac{9}{2}\right)$$
$$= \frac{9}{2}\cdot\Gamma\left(\frac{7}{2}+1\right)$$
$$= \frac{9}{2}\cdot\frac{7}{2}\Gamma\left(\frac{7}{2}\right)$$
$$= \frac{9}{2}\cdot\frac{7}{2}\cdot\Gamma\left(\frac{5}{2}+1\right)$$
$$= \frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\Gamma\left(\frac{5}{2}\right)$$
$$= \frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\,\Gamma\left(\frac{3}{2}+1\right)$$
$$= \frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right)$$
$$= \frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\Gamma\left(\frac{1}{2}+1\right)$$
$$= \frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$
$$= \frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$$